24 The finite element method

24.1 Introduction

In this chapter the finite element method proper[10] will be described with the aid of worked examples.

The finite element method is based on the matrix displacement method described in Chapter 23, but its description is separated from that chapter because it can be used for analysing much more complex structures, such as those varying from the legs of an integrated circuit to the legs of an offshore drilling rig, or from a gravity dam to a doubly curved shell roof. Additionally, the method can be used for problems in structural dynamics, fluid flow, heat transfer, acoustics, magnetostatics, electrostatics, medicine, weather forecasting, etc.

The method is based on representing a complex shape by a series of simpler shapes, as shown in Figure 24.1, where the simpler shapes are called finite elements.

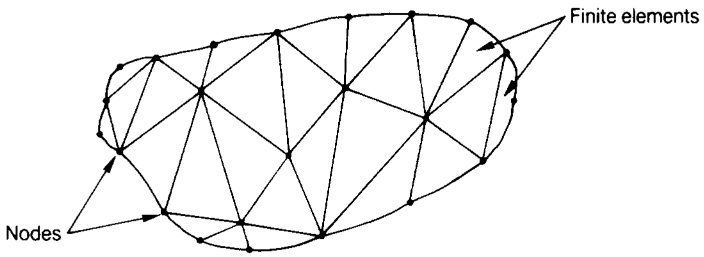


Figure 24.1 Complex shape, represented by finite elements.

Using the energy methods described in Chapter 17, the stiffness and other properties of the finite element can be obtained, and then by considering equilibrium and compatibility along the inter-element boundaries, the stiffness and other properties of the entire domain can be obtained.

[10]Turner M J, Clough R W, Martin H C and Topp L J, Stiffness and Deflection Analysis of Complex Structures, *J Aero. Sci*, 23, 805–23, 1956.

This process leads to a large number of simultaneous equations, which can readily be solved on a high-speed digital computer. It must be emphasised, however, that the finite element method is virtually useless without the aid of a computer, and this is the reason why the finite element method has been developed alongside the advances made with digital computers. Today, it is possible to solve massive problems on most computers, including microcomputers, laptop and notepad computers; and in the near future, it will be possible to use the finite element method with the aid of hand-held computers.

Finite elements appear in many forms, from triangles and quadrilaterals for two-dimensional domains to tetrahedrons and bricks for three-dimensional domains, where, in general, the finite element is used as a 'space' filler.

Each finite element is described by nodes, and the nodes are also used to describe the domain, as shown in Figure 24.1, where corner nodes have been used.

If, however, mid-side nodes are used in addition to corner nodes, it is possible to develop curved finite elements, as shown in Figure 24.2, where it is also shown how ring nodes can be used for axisymmetric structures, such as conical shells.

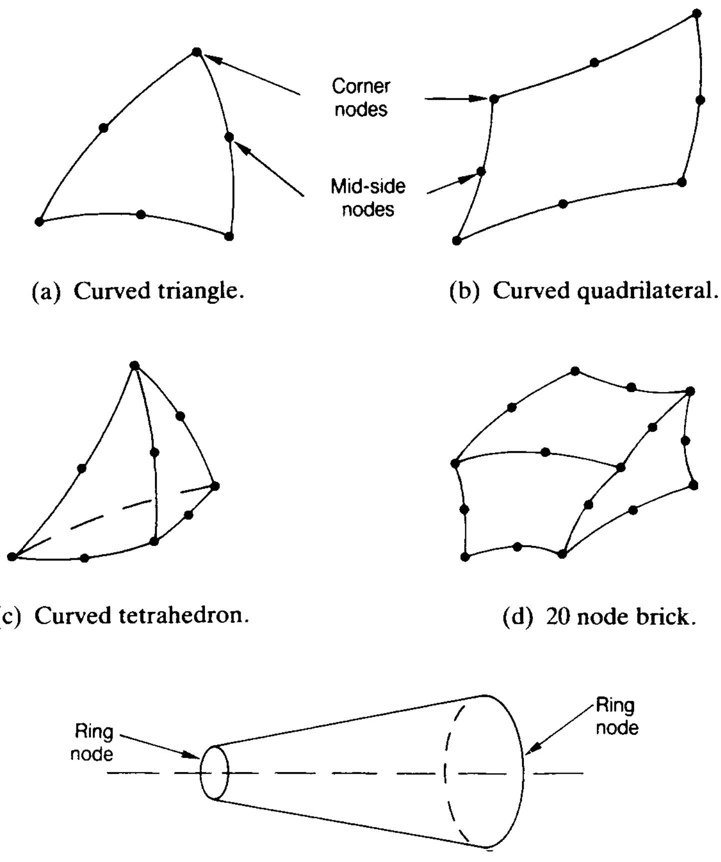


Figure 24.2 Some typical finite elements.

The finite element was invented in 1956 by Turner *et al.* where the important three node in-plane triangular finite element was first presented.

The derivation of the stiffness matrix for this element will now be described.

24.2 Stiffness matrices for some typical finite elements

The in-plane triangular element of Turner *et al.* is shown in Figure 24.3. From this figure, it can be seen that the element has six degrees of freedom, namely, u_1°, u_2°, u_3°, v_1°, v_2° and v_3°, and because of this, the assumptions for the displacement polynomial distributions u° and v° will involve six arbitrary constants. It is evident that with six degrees of freedom, a total of six simultaneous equations will be obtained for the element, so that expressions for the six arbitrary constants can be defined in terms of the nodal displacements, or boundary values.

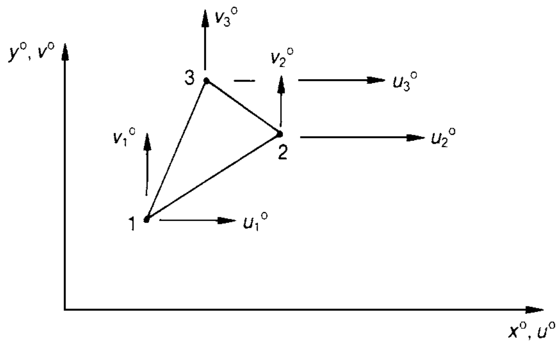


Figure 24.3 In-plane triangular element.

Convenient displacement equations are

$$u^\circ = \alpha_1 + \alpha_2 x^\circ + \alpha_3 y^\circ \tag{24.1}$$

and

$$v^\circ = \alpha_4 + \alpha_5 x^\circ + \alpha_6 y^\circ \tag{24.2}$$

where α_1 to α_6 are the six arbitrary constants, and u° and v° are the displacement equations.

Suitable boundary conditions, or boundary values, at node 1 are:

$$\text{at } x^\circ = x_1^\circ \text{ and } y^\circ = y_1^\circ, \quad u^\circ = u_1^\circ \text{ and } v^\circ = v_1^\circ$$

Substituting these boundary values into equations (24.1) and (24.2),

$$u_1^\circ = \alpha_1 + \alpha_2 x_1^\circ + \alpha_3 y_1^\circ \tag{24.3}$$

and

$$v_1^\circ = \alpha_4 + \alpha_5 x_1^\circ + \alpha_6 y_1^\circ \tag{24.4}$$

Similarly, at node 2,

$$\text{at } x^\circ = x_2^\circ \text{ and } y^\circ = y_2^\circ, \quad u^\circ = u_2^\circ \text{ and } v^\circ = v_2^\circ$$

When substituted into equations (24.1) and (24.2), these give

$$u_2^\circ = \alpha_1 + \alpha_2 x_2^\circ + \alpha_3 y_2^\circ \tag{24.5}$$

and

$$v_2^\circ = \alpha_4 + \alpha_5 x_2^\circ + \alpha_6 y_2^\circ \tag{24.6}$$

Likewise, at node 3,

$$\text{at } x^\circ = x_3^\circ \text{ and } y^\circ = y_3^\circ, \quad u^\circ = u_3^\circ \text{ and } v^\circ = v_3^\circ$$

which, when substituted into equation (24.1) and (24.2), yield

$$u_3^\circ = \alpha_1 + \alpha_2 x_3^\circ + \alpha_3 y_3^\circ \tag{24.7}$$

and

$$v_3^\circ = \alpha_4 + \alpha_5 x_3^\circ + \alpha_6 y_3^\circ \tag{24.8}$$

Rewriting equations (24.3) to (24.8) in matrix form, the following equation is obtained:

$$\begin{Bmatrix} u_1^\circ \\ u_2^\circ \\ u_3^\circ \\ v_1^\circ \\ v_2^\circ \\ v_3^\circ \end{Bmatrix} = \begin{bmatrix} 1 & x_1^\circ & y_1^\circ & & & \\ 1 & x_2^\circ & y_2^\circ & & 0_3 & \\ 1 & x_3^\circ & y_3^\circ & & & \\ & & & 1 & x_1^\circ & y_1^\circ \\ & 0_3 & & 1 & x_2^\circ & y_2^\circ \\ & & & 1 & x_3^\circ & y_3^\circ \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \tag{24.9}$$

or

$$\{u_1^\circ\} = \begin{bmatrix} A & 0_3 \\ 0_3 & A \end{bmatrix} \{a_i\} \tag{24.10}$$

and

$$\{a_i\} = \begin{bmatrix} A^{-1} & 0_3 \\ 0_3 & A^{-1} \end{bmatrix} \begin{Bmatrix} u_1^\circ \\ u_2^\circ \\ u_3^\circ \\ v_1^\circ \\ v_2^\circ \\ v_3^\circ \end{Bmatrix} \tag{24.11}$$

where

$$[\mathbf{A}]^{-1} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} / \ \mathbf{det}|\mathbf{A}| \tag{24.12}$$

$$\begin{aligned} a_1 &= x_2^\circ y_3^\circ - x_3^\circ y_2^\circ \\ a_2 &= x_3^\circ y_1^\circ - x_1^\circ y_3^\circ \\ a_3 &= x_1^\circ y_2^\circ - x_2^\circ y_1^\circ \\ b_1 &= y_2^\circ - y_3^\circ \\ b_2 &= y_3^\circ - y_1^\circ \\ b_3 &= y_1^\circ - y_2^\circ \\ c_1 &= x_3^\circ - x_2^\circ \\ c_2 &= x_1^\circ - x_3^\circ \\ c_3 &= x_2^\circ - x_1^\circ \end{aligned} \tag{24.13}$$

$$\mathbf{det}\ |A| = x_2^\circ y_3^\circ - y_2^\circ x_3^\circ - x_1^\circ (y_3^\circ - y_2^\circ) + y_1^\circ (x_3^\circ - x_2^\circ) = 2\Delta$$

Δ = area of triangle

Substituting equations (24.13) and (24.12) into equations (24.1) and (24.2)

$$\begin{Bmatrix} u^\circ \\ v^\circ \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1^\circ \\ u_2^\circ \\ u_3^\circ \\ v_1^\circ \\ v_2^\circ \\ v_3^\circ \end{Bmatrix} \tag{24.14}$$

or

$$\{u\} = [\mathbf{N}]\,\{u_i\} \tag{24.15}$$

where $[\mathbf{N}]$ = a matrix of shape functions:

$$\begin{aligned} N_1 &= \frac{1}{2\Delta}\left(a_1 + b_1 x^\circ + c_1 y^\circ\right) \\ N_2 &= \frac{1}{2\Delta}\left(a_2 + b_2 x^\circ + c_2 y^\circ\right) \\ N_3 &= \frac{1}{2\Delta}\left(a_3 + b_3 x^\circ + c_3 y^\circ\right) \end{aligned} \tag{24.16}$$

For a two-dimensional system of strain, the expressions for strain[11] are given by

$$\begin{aligned} \varepsilon_x &= \text{strain in the } x^\circ \text{ direction} = \partial u^\circ/\partial x^\circ \\ \varepsilon_y &= \text{strain in the } y^\circ \text{ direction} = \partial v^\circ/\partial y^\circ \\ \gamma_{xy} &= \text{shear strain in the } x^\circ\text{–}y^\circ \text{ plane} \\ &= \partial u^\circ/\partial y^\circ + \partial v^\circ/\partial x^\circ \end{aligned} \tag{24.17}$$

which when applied to equation (24.14) becomes

[11] Fenner R T, *Engineering Elasticity,* Ellis Horwood, 1986.

$$2\Delta\varepsilon_x = b_1 u_1{}^\circ + b_2 u_2{}^\circ + b_3 u_3{}^\circ$$

$$2\Delta\varepsilon_y = c_1 v_1{}^\circ + c_2 v_2{}^\circ + c_3 v_3{}^\circ$$

$$2\Delta\gamma_{xy} = c_1 u_1{}^\circ + c_2 u_2{}^\circ + c_3 u_3{}^\circ + b_1 v_1{}^\circ + b_2 v_2{}^\circ + b_3 v_3{}^\circ \tag{24.18}$$

Rewriting equation (24.18) in matrix form, the following is obtained:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_1 & b_2 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & b_1 & b_2 & b_3 \end{bmatrix} \begin{Bmatrix} u_1{}^\circ \\ u_2{}^\circ \\ u_3{}^\circ \\ v_1{}^\circ \\ v_2{}^\circ \\ v_3{}^\circ \end{Bmatrix} \tag{24.19}$$

or

$$\{\varepsilon\} = [\mathbf{B}]\,\{u_i\} \tag{24.20}$$

where [**B**] is a matrix relating strains and nodal displacements

$$[\mathbf{B}] = \frac{1}{2\Delta} \begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & b_1 & b_2 & b_3 \end{bmatrix} \tag{24.21}$$

Now, from Chapter 5, the relationship between stress and strain for plane stress is given by

$$\sigma_x = \frac{E}{(1 - v^2)}(\varepsilon_x + v\varepsilon_y)$$

$$\sigma_y = \frac{E}{(1 - v^2)}(v\varepsilon_x + \varepsilon_y) \tag{24.22}$$

$$\tau_{xy} = \frac{E}{2(1 + v)}\gamma_{xy}$$

where

σ_x = direct stress in the $x°$-direction

σ_y = direct stress in the $y°$-direction

τ_{xy} = shear stress in the $x°$–$y°$ plane

E = Young's modulus of elasticity

ν = Poisson's ratio

ε_x = direct strain in the $x°$-direction

ε_y = direct strain in the $y°$-direction

γ_{xy} = shear strain in the $x°$–$y°$ plane

$$G = \text{shear modulus} = \frac{E}{2(1+\nu)}$$

Rewriting equation (24.22) in matrix form,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{24.23}$$

or

$$\{\sigma\} = [\mathbf{D}]\{\varepsilon\} \tag{24.24}$$

where, for plane stress,

$$[\mathbf{D}] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{24.25}$$

= a matrix of material constants

and for plane strain,[12]

$$[\mathrm{D}] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \tag{24.26}$$

or, in general,

$$[\mathbf{D}] = E^1 \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \gamma \end{bmatrix} \tag{24.27}$$

where, for plane stress,

$$E^1 = E/(1 - \nu^2)$$

$$\mu = \nu$$

$$\gamma = (1 - \nu)/2$$

and for plane strain,

$$E^1 = E(1 - \nu)/[(1 + \nu)(1 - 2\nu)]$$

$$\mu = \nu/(1 - \nu)$$

$$\gamma = (1 - 2\nu)/[2(1 - \nu)]$$

Now from Section 1.13, it can be seen that the general expression for the strain energy of an elastic system, U_e, is given by

$$U_e = \int \frac{\sigma^2}{2E}\, d(\mathrm{vol})$$

but

$$\sigma = E\varepsilon$$

$$\therefore\ U_e = \frac{1}{2}\int E\varepsilon^2\, d(\mathrm{vol})$$

---

[12] Ross, C T F, *Mechanics of Solids*, Prentice Hall, 1996.

which, in matrix form, becomes

$$U_e = \frac{1}{2} \int \{\varepsilon\}^T [\mathbf{D}] \{\varepsilon\} d\,(\text{vol}) \tag{24.28}$$

where,

$\{\varepsilon\}$ = a vector of strains, which for this problem is

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{24.29}$$

$[\mathbf{D}]$ = a matrix of material constants

It must be remembered that U_e is a scalar and, for this reason, the vector and matrix multiplication of equation (24.28) must be carried out in the manner shown.

Now, the work done by the nodal forces is

$$\text{WD} = -\{u_i^\circ\}^T \{P_i^\circ\} \tag{24.30}$$

where $\{P_i^\circ\}$ is a vector of nodal forces

and the total potential is

$$\begin{aligned} \pi_p &= U_e + \text{WD} \\ &= \frac{1}{2} \int \{\varepsilon\}^T [\mathbf{D}] \{\varepsilon\}\ d(\text{vol}) - \{u_i^\circ\}\ \{P_i^\circ\} \end{aligned} \tag{24.31}$$

It must be remembered that WD is a scalar and, for this reason, the premultiplying vector must be a row vector, and the postmultiplying vector must be a column vector.

Substituting equation (24.20) into (24.31):

$$\pi_p = \frac{1}{2}\{u_1^\circ\}^T \int [\mathrm{B}]^T [\mathrm{D}][\mathrm{B}] d(\text{vol}) \{u_i^\circ\} - \{u_i^\circ\}^T \{P_i^\circ\} \tag{24.32}$$

but according to the method of minimum potential (see Chapter 17),

$$\frac{\partial \pi_p}{\partial \{u_i^\circ\}} = 0$$

or

$$0 = \int [\mathbf{B}]^{\mathrm{T}} [\mathbf{D}] [\mathbf{B}]\, d(\mathrm{vol}) \{u_i^\circ\} - \{P_i^\circ\}$$

i.e.

$$\{P_i^\circ\} = \int [\mathbf{B}]^{\mathrm{T}} [\mathbf{D}] [\mathbf{B}]\, d(\mathrm{vol}) \{u_i^\circ\} \tag{24.33}$$

but,

$$\{P_i^\circ\} = [\mathbf{k}^\circ] \{u_i^\circ\}$$

or,

$$[\mathbf{k}^\circ] = \int [\mathbf{B}]^{\mathrm{T}} [\mathbf{D}] [\mathbf{B}]\, d(\mathrm{vol}) \tag{24.34}$$

Substituting equations (24.21) and (24.27) into equation (24.34):

$$[\mathbf{k}^\circ] = t \begin{bmatrix} P_{ij} & Q_{ij} \\ Q_{ji} & R_{ij} \end{bmatrix} \tag{24.35}$$

$$P_{ij} = 0.25\, E^l\, (b_i b_j + \gamma c_i c_j)/\Delta$$

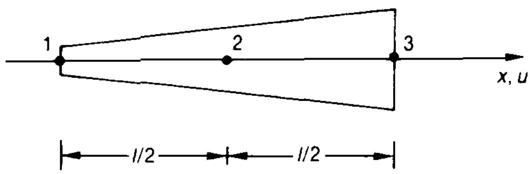
$$Q_{ij} = 0.25\, E^l\, (\mu b_i c_j + \gamma c_i b_j)/\Delta$$

$$Q_{ji} = 0.25\, E^l\, (\mu b_j c_i + \gamma c_j b_i)/\Delta$$

$$R_{ij} = 0.25\, E^l\, (c_i c_j + \gamma b_i b_j)/\Delta \tag{24.36}$$

where i and j vary from 1 to 3 and t is the plate thickness

Problem 24.1 Working from first principles, determine the elemental stiffness matrix for a rod element, whose cross-sectional area varies linearly with length. The element is described by three nodes, one at each end and one at mid-length, as shown below. The cross-sectional area at node 1 is A and the cross-sectional area at node 3 is $2A$.



<u>Solution</u>

As there are three degrees of freedom, namely u_1, u_2 and u_3, it will be convenient to assume a polynomial involving three arbitrary constants, as shown by equation (24.37):

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{24.37}$$

To obtain the three simultaneous equations, it will be necessary to assume the following three boundary conditions or boundary values:

$$\begin{aligned} &\text{At } x = 0,\ u = u_1 \\ &\text{At } x = l/2,\ u = u_2 \\ &\text{At } x = l,\ u = u_3 \end{aligned} \tag{24.38}$$

Substituting equations (24.38) into equation (24.37), the following three simultaneous equations will be obtained:

$$u_1 = \alpha_1 \tag{24.39a}$$

$$u_2 = \alpha_1 + \alpha_2 l/2 + \alpha_3 l^2/4 \tag{24.39b}$$

$$u_3 = \alpha_1 + \alpha_2 l + \alpha_3 l^2 \tag{24.39c}$$

From (24.39a)

$$\alpha_1 = u_1 \tag{24.40}$$

Dividing (24.39c) by 2 gives

$$u_3/2 = u_1/2 + \alpha_2 l/2 + \alpha_3 l^2/2 \tag{24.41}$$

Taking (24.41) from (24.39b),

$$u_2 - u_3/2 = u_1 - u_1/2 - \alpha_3 l^2/4$$

or

$$\alpha_3 \frac{l^2}{4} = u_1/2 - u_2 + u_3/2$$

$$\alpha_3 = \frac{1}{l^2}(2u_1 - 4u_2 + 2u_3) \tag{24.42}$$

Substituting equations (24.40) and (24.42) into equation (24.39c),

$$\alpha_2 l = u_3 - u_1 - 2u_1 + 4u_2 - 2u_3$$

or

$$\alpha_2 = \frac{1}{l}(-3u_1 + 4u_2 - u_3) \tag{24.43}$$

Substituting equations (24.40), (24.42) and (24.43) into equation (24.37),

$$u = u_1 + \left(-3u_1 + 4u_2 - u_3\right)\xi + \left(2u_1 - 4u_2 + 2u_3\right)\xi^2$$

$$= u_1\left(1 - 3\xi + 2\xi^2\right) + u_2\left(4\xi - 4\xi^2\right) + u_3\left(-\xi + 2\xi^2\right)$$

$$u = \left[\left(1 - 3\xi + 2\xi^2\right)\left(4\xi - 4\xi^2\right)\left(-\xi + 2\xi^2\right)\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [\mathbf{N}]\{u_i\}$$

where

$$\xi = x/l$$

$$[\mathbf{N}] = \left[\left(1 - 3\xi + 2\xi^2\right)\ \left(4\xi - 4\xi^2\right)\ \left(-\xi + 2\xi^2\right)\right] \tag{24.44}$$

Now,

$$\varepsilon = \frac{du}{dx} = \frac{du}{l\,d\xi}$$

$$= \frac{1}{l}\left[(-3 + 4\xi)\ (4 - 8\xi)\ (-1 + 4\xi)\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [\mathbf{B}]\ \{u_i\}$$

or

$$[\mathbf{B}] = \frac{1}{l}\left[(-3 + 4\xi)\ (4 - 8\xi)\ (-1 + 4\xi)\right] \tag{24.45}$$

Now, for a rod,

$$\frac{\sigma}{\varepsilon} = E$$

or

$$\sigma = E\varepsilon$$

$$\therefore [\mathbf{D}] = E$$

Now.

$$[\mathbf{k}] = \int [\mathbf{B}]^{\mathrm{T}}\ [\mathbf{D}]\ [\mathbf{B}]\ d(\mathrm{vol})$$

$$= \int [\mathbf{B}]^{\mathrm{T}}\ [\mathbf{D}]\ [\mathbf{B}]\ al\ d\xi$$

where

$$a = \text{area at } \xi = A(1 + \xi)$$

$$\therefore [\mathbf{k}] = \int_0^{\xi} \frac{1}{l^2}\left[\begin{pmatrix} -3 + 4\xi \\ 4 - 8\xi \\ -1 + 4\xi \end{pmatrix}\right]$$

$$\times\ E\left[(-3 + 4\xi)(4 - 8\xi)(-1 + 4\xi)\right]\ A(1 + \xi)\ l\,d\xi$$

$$= \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

where,

$$k_{11} = \frac{AE}{l} \int_0^1 (-3 + 4\xi)^2 (1 + \xi) \, d\xi$$

$$= 2.8333 \; AE/l$$

$$k_{22} = \frac{AE}{l} \int_0^1 (4 - 8\xi)^2 (1 + \xi) d\xi$$

$$= 8 \; AE/l$$

$$k_{33} = \frac{AE}{l} \int_0^1 (-1 + 4\xi)^2 (1 + \xi) \, d\xi$$

$$= 4.167 \; AE/l$$

$$k_{12} = k_{21} = \frac{AE}{l} \int_0^1 (-3 + 4\xi) (4 - 8\xi) (1 + \xi) d\xi$$

$$= -3.33 AE/l$$

$$k_{13} = k_{31} = \frac{AE}{l} \int_0^1 (-3 + 4\xi) (-1 + 4\xi) (1 + \xi) d\xi$$

$$= AE/2l$$

$$k_{23} = k_{32} = \frac{AE}{l} \int_0^1 (4 - 8\xi) (-1 + 4\xi) (1 + \xi) \, d\xi$$

$$= -4.667 \; AE/l$$

In this chapter, it has only been possible to introduce the finite element method, and for more advanced work on this topic, the reader is referred to Ross, C T F, *Advanced Applied Finite Element Methods*, Ellis Horword; Zienkiewicz, O C, and Taylor, R L. *The Finite Element Method*, McGraw-Hill, Vol 1, 1989, Vol 2, 1991.

Further problems *(answers on page 698)*

24.2 Using equation (24.34), determine the stiffness matrix for a uniform section rod element, with two degrees of freedom.

24.3 A rod element has a cross-sectional area which varies linearly from A_1 at node 1 to A_2 at node 2, where the nodes are at the ends of the rod. If the rod element has two degrees of freedom, determine its elemental stiffness matrix using equation (24.34).

24.4 Using equation (24.34), determine the stiffness matrix for a uniform section torque bar which has two degrees of freedom.

24.5 Using equation (24.34), determine the stiffness matrix for a two node uniform section beam, which has four degrees of freedom; two rotational and two translational.